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Simultaneous control of longitudinal and transverse vibrations of an axially moving string with velocity tracking

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ABSTRACT

In this paper, an active control scheme for an axially moving string system that suppresses both longitudinal and transverse vibrations and regulates the transport velocity of the string to track a desired moving velocity profile is investigated. The control scheme utilizes three inputs: one control force at the right boundary, which is exerted by a hydraulic actuator equipped with a damper, and two control torques applied at the left and right rollers. The equations of motion are derived by using Hamilton's principle. Two nonlinear partial differential equations govern the longitudinal and transverse motions, where the variation of the tension of the string due to the transverse and longitudinal vibrations is considered. Among four boundary conditions, two describe the rotational dynamics of the left and right rollers; one determines the dynamics of the hydraulic actuator at the right boundary, and the last one denotes that the left boundary is fixed. The Lyapunov method is employed to generate control laws. Asymptotic stability of the transverse and longitudinal dynamics and the velocity tracking error is achieved. The effectiveness of the proposed control scheme is illustrated via numerical simulations.

1. Introduction

There are numerous industries that use axially moving systems such as papers, textiles, metal sheets, polymers, and composite materials. Application of these systems yields better performance and supports mass production and high-speed automation. However, mechanical vibration of the moving material (in both longitudinal and transverse directions), especially in high-speed precision machine systems, becomes the main quality- and productivity-limiting factor. Such vibration must be suppressed, and in many industries, the transport velocity of the moving material is required to be controlled in order to track time-varying velocity as well as constant velocity profiles. Moreover, it is well known that variations of transport velocity can result in vibrations of the moving material. Therefore, the prompt transverse and longitudinal vibration suppression together with the transport velocity control is desirable for axially moving material systems.

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To solve the transverse vibration control problems of axially moving systems [1–7] and of flexible string/beam systems [8–12], many researchers have investigated the application of control actions at the left or right boundary [13–21] (a measure known as boundary control), because the provision of control inputs through a supporting roller is more cost-effective than the addition of an extra actuator at a middle point in the system. These achievements were all predicated on the Lyapunov method, by which control laws to reduce vibration energy to zero are derived using Lyapunov function candidates based on the total mechanical energy of the moving system. These control laws incorporate the measured signal of the transverse displacement along with the time rate of the slope of the moving material at the boundaries, both obtainable via laser sensors at the boundary points [16]. Therefore, in so far as actuators and sensors can easily be assembled at the boundaries, the boundary control method can be a practical control solution for axially moving material systems. Fung et al. [15] developed a boundary control scheme for an axially moving string, in which an adaptive boundary control law was applied to a mass–damper–spring mechanism to suppress transverse vibration. Li et al. [16] introduced an adaptive isolation scheme for an axially moving string divided into two spans by a transverse force actuator in order to reduce the transverse vibration of the controlled span to zero under bounded disturbances in the uncontrolled span. Nguyen and Hong [21] proposed two schemes of robust adaptive boundary control for an axially moving string of unknown system parameters under spatially varying tension and unknown boundary disturbance.

In recent years, there have been many papers published on velocity and tension control of axially moving material systems [21–27]. These studies considered only the longitudinal dynamics of the moving material as represented by either ordinary differential equations [22–25] or partial differential equations (PDEs: [26,27]). Koc et al. [22] developed an H-infinity robust control strategy with varying control gains for unwind/rewind sections under the radius variations of unwind/rewind rollers. Pagilla et al. [23] proposed a decentralized control scheme for a web processing line, based on a dynamic model taking into account the radial and inertial variations of unwind/rewind rollers. Zhao and Rahn [27] applied an iterative learning control algorithm to an axially moving string system, enabling precise tension and velocity regulation.

Thus far, little research has been devoted to investigating the vibration control of an axially moving system together with its transport velocity control. Nagarkatti et al. [3] was the first paper to study coupled vibration and velocity control for an axially moving string wherein axial velocity was regulated by two control torques driving rollers at the boundaries, and where transverse vibration was suppressed via an actuator in the middle of the web span. In this investigation, however, the longitudinal motion of the string was neglected. It should be noted that the torque used to regulate the axial velocity affects not only the material tension but also the longitudinal motion as well and, consequently, the transverse motion [28]. Therefore, for coupled vibration suppression and transport velocity control, the control scheme should be designed to suppress both longitudinal and transverse vibrations. On this premise, we developed an active control scheme that suppresses both the longitudinal and transverse vibrations of an axially moving string system and regulates its transport velocity in order to track a desired profile. The scheme incorporates a hydraulic actuator equipped with a damper at the right boundary and applies separate torques to the left and right rollers, as shown in Fig. 1.

Contributions of this paper are the following. First, a dynamic model including the longitudinal and transverse dynamics and three actuators dynamics (one hydraulic actuator and two rollers) was derived using the Hamilton's principle, where the material tension (spatially varying tension) is considered as a nonlinear function of the string slope and the change in length of the longitudinal displacement. Second, a control scheme for simultaneous suppression of longitudinal and transverse vibrations and velocity regulation was designed using the Lyapunov method. The scheme consists of three control laws that generate the required signals for the force exerted by the hydraulic actuator and the torques at the left and right rollers. Third, the scheme guarantees the asymptotic convergence of the longitudinal and transverse vibrations and velocity tracking error to zero.

The rest of this paper is organized as follows. Section 2 presents the dynamic model of the considered system (an axially moving viscoelastic string system), in which two nonlinear PDEs govern the longitudinal and transverse displacements, respectively, and the boundary conditions determine the dynamics of the actuators. Section 3 introduces the proposed control scheme design, wherein the spatially varying tension is considered explicitly. A Lyapunov function-based stability analysis of the closed-loop system and the proof of asymptotic stability also are discussed. Section 4 includes numerical simulation results that illustrate the effectiveness of the proposed control scheme. Finally, Section 5 draws conclusions.

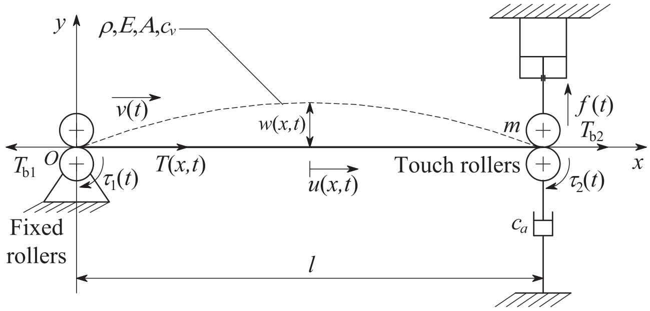

Fig. 1. Schematic of an axially moving string system driven by two rollers with a hydraulic actuator at the right boundary.

2. Problem formulation

Fig. 1 shows a schematic of the axially moving string system driven by two rollers at the boundaries, which are used to control the longitudinal vibration. For transverse vibration suppression, a control mechanism that includes a hydraulic actuator and a damper is located at the right boundary. The left boundary is fixed, restricting the movement of the string in the vertical direction. Conversely, at the right boundary, the control mechanism allows for transverse (vertical) movement of the string in accordance with the hydraulic actuator dynamics.

The axially moving string travels between the two rollers at a time-varying transport velocity $v(t)$ in the x-direction. Let l be the distance between the two rollers, A the cross-sectional area, ρ the mass per unit length, c_v the viscous damping coefficient, and E the Young's modulus. The longitudinal displacement $u(x,t)$ and the transverse displacement $w(x,t)$ represent the motions of the string in the fixed inertial frame Oxy. For notational convenience, instead of $u_x(x,t)$ and $u_t(x,t)$, u_x and u_t are used, and similar abbreviations are employed subsequently.

The kinetic energy of the axially moving string is represented as

$$K = \frac{1}{2}\int_0^l \rho\{(v+u_t+vu_x)^2+(w_t+vw_x)^2\}\,dx+\frac{1}{2}mw_t^2(l,t)+\frac{J}{2R^2}\{u_t^2(0,t)+u_t^2(l,t)\}, \tag{1}$$

where m is the lumped mass of the hydraulic actuator, and J and R denote the moment of inertia and the radius of the two rollers, respectively (the two rollers are assumed to be exactly the same). The potential energy is obtained as

$$U = \frac{1}{2}\int_0^l T\varepsilon\,dx \tag{2}$$

The spatially varying tension T in Eq. (2) is given as [28–30]

$$T = T_0+EA(u_x+w_x^2/2), \tag{3}$$

where T_0 is the tension of the undisturbed string. The displacement–strain relation ε is expressed as

$$\varepsilon = u_x+w_x^2/2. \tag{4}$$

The virtual work done by the non-conservative forces is computed as

$$\begin{aligned}\delta W &= (\tau_1(t)/R-T_{b1})\delta u(0,t)+(\tau_2(t)/R+T_{b2})\delta u(l,t)+f(t)\delta w(l,t)\\ &\quad-\int_0^l c_v(w_t+vw_x)\delta w\,dx+c_a w_t(l,t)\delta w(l,t),\end{aligned} \tag{5}$$

where the material tensions in the respective adjacent spans (T_{b1} and T_{b2}) are assumed to be constant, $\tau_1(t)$ and $\tau_2(t)$ are the control torques applied to the two drive rollers, $f(t)$ is the control force exerted by the hydraulic actuator, and c_a is the damping coefficient of the damper (Fig. 1). The virtual momentum transport across the boundaries is given as

$$\delta M = \rho v(\mathbf{v}\cdot\delta\mathbf{r})\big|_0^l, \tag{6}$$

where the displacement vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ are given as

$$\mathbf{r} = (x+u)\mathbf{i}+w\mathbf{j}, \tag{7}$$

$$\mathbf{v} = (v+u_t+vu_x)\mathbf{i}+(w_t+vw_x)\mathbf{j}, \tag{8}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and y-directions, respectively.

Using the extended Hamilton's principle [31–34], the governing equations of motion for the axially moving string system with longitudinal and transverse motions, respectively, are derived as

$$\rho(u_{tt}+\dot{v}u_x+2vu_{xt}+v^2u_{xx}+\dot{v})-(EA(u_x+w_x^2/2))_x = 0, \tag{9}$$

$$\rho(w_{tt}+\dot{v}w_x+2vw_{xt}+v^2w_{xx})+c_v(w_t+vw_x)-((T_0+EA(u_x+w_x^2/2))w_x)_x = 0. \tag{10}$$

The boundary conditions are obtained as

$$\frac{J}{R^2}u_{tt}(0,t)-EA\left(u_x(0,t)+\frac{w_x^2(0,t)}{2}\right) = \frac{\tau_1(t)}{R}-T_{b1}+T_0, \tag{11}$$

$$\frac{J}{R^2}u_{tt}(l,t)+EA\left(u_x(l,t)+\frac{w_x^2(l,t)}{2}\right) = \frac{\tau_2(t)}{R}+T_{b2}-T_0, \tag{12}$$

$$w(0,t) = 0, \tag{13}$$

$$mw_{tt}(l,t)+[T_0+EA(u_x(l,t)+w_x^2(l,t)/2)]w_x(l,t)+(c_a-\rho v)w_t(l,t)-\rho v^2 w_x(l,t) = f(t). \tag{14}$$

The initial conditions are given as

$$u(x,0) = g_1(x),\quad u_t(x,0) = g_2(x),\quad w(x,0) = h_1(x),\quad w_t(x,0) = h_2(x). \tag{15}$$

The relationship between the torques and the transport velocity is given as

$$\left(\rho lR+\frac{2J}{R}\right)\dot{v}(t)=\tau_1(t)+\tau_2(t)+(T_{b2}-T_{b1})R. \tag{16}$$

Remark 1. As shown in Eqs. (9)–(14), the actuators are coupled with the string system, where Eqs. (11) and (12) determine the dynamics of the two drive rollers in compliance with the torques $\tau_1(t)$ and $\tau_2(t)$, and Eq. (14) represents the dynamics of the hydraulic actuator in compliance with the force $f(t)$. Therefore, to achieve the stability of the coupled system Eqs. (9)–(14), not only the convergence of the longitudinal and transverse displacements to zero needs to be obtained, but the convergence of the motions of the actuators to zero must also be satisfied.

Remark 2. The torques $\tau_1(t)$ and $\tau_2(t)$ affect not only the transport velocity $v(t)$ but also longitudinal displacement u as well as the material tension T, as shown in the tension formula (3), the boundary conditions (11) and (12), and the transport velocity dynamics (16). Consequently, the longitudinal displacement can change the transverse dynamics of the string through the change of the longitudinal displacement u_x, as shown in Eq. (10). Therefore, when vibration suppression and velocity control are coupled, the control scheme has to be designed in consideration of both the longitudinal and transverse dynamics.

3. Control formulation

The control objective is to suppress the longitudinal and transverse vibrations and to regulate the transport velocity to track a desired profile. Based on the Lyapunov method, a control scheme employing control force $f(t)$ and control torques $\tau_1(t)$ and $\tau_2(t)$ is designed to achieve the asymptotic convergence of all the longitudinal and transverse vibrations and the velocity tracking error to zero.

The velocity tracking error is defined as

$$e(t)=v(t)-v_d(t), \tag{17}$$

where $v_d(t)$ is the desired velocity. Using Eq. (17), Eq. (16) is then rewritten as

$$\left(\rho lR+\frac{2J}{R}\right)\dot{e}(t)=\tau_1(t)+\tau_2(t)+(T_{b2}-T_{b1})R-\dot{v}_d(t)\left(\rho lR+\frac{2J}{R}\right). \tag{18}$$

The desired transport velocity of the string, its time derivative, and the initial velocity tracking error are assumed to be bounded as follows:

$$|v_d(t)|\le\xi_1,\quad |\dot{v}_d(t)|\le\xi_2,\ e(0)|\le\xi_3, \tag{19}$$

where ξ_i (i=1, 2, 3) is a positive constant.

Based on the total mechanical energy of the string, the following function is introduced:

$$V(t)=\alpha\left(V_0(t)+\frac{1}{2}\int_0^l EAu_xw_x^2\,dx\right)+\beta V_1(t), \tag{20}$$

where α and β are positive constants, $V_0(t)$ is defined as

$$\begin{aligned}V_0(t)=&\frac{1}{2}\int_0^l\rho(v+u_t+vu_x)^2\,dx+\frac{1}{2}\int_0^l\rho(w_t+vw_x)^2\,dx+\frac{1}{2}\int_0^l EAu_x^2\,dx+\frac{1}{2}\int_0^l T_0w_x^2\,dx+\frac{1}{8}\int_0^l EAw_x^4\,dx+\frac{EA}{4l}u^2(l,t)\\&+\frac{J}{2R^2}(u_t^2(0,t)+u_t^2(l,t))+\frac{1}{2}m\left[\left(w_t(l,t)+\left(v+\frac{2\beta l}{\alpha}\right)w_x(l,t)\right)^2+w_x^2(l,t)\right]+\frac{1}{2}e^2(t),\end{aligned} \tag{21}$$

and $V_1(t)$ is given as

$$V_1(t)=2\rho\left(\int_0^l x\{u_x(v+u_t+vu_x)+w_x(w_t+vw_x)\}\,dx\right). \tag{22}$$

Now, the following control laws are proposed:

$$f(t)=-k_{f1}(t)w_t(l,t)+k_{f2}(t)w_x(l,t)-\rho v^2w_x(l,t)+(c_a-\rho v)w_t(l,t)-(v+2\beta l/\alpha)mw_{xt}(l,t), \tag{23}$$

$$\begin{aligned}\tau_1(t)=\frac{EAR}{\alpha v}\Bigg\{&\frac{\alpha v(T_{b1}-T_0)}{EA}+\beta lEAu_x(l,t)w_x^2(l,t)-\frac{\beta EA}{2l}\left(1-\frac{1}{2k_{\tau1}}\right)u^2(l,t)-\frac{(u_t^2(0,t)+u_t^2(l,t))}{k_{\tau1}}-\alpha(v+u_t(l,t)+vu_x(l,t))(u_x(l,t)\\&+w_x^2(l,t)/2)+\alpha(u_t(0,t)+vu_x(0,t))(u_x(0,t)+w_x^2(0,t)/2)-\frac{EA\alpha}{2l}u(l,t)u_t(l,t)-\beta EAlu_x^2(l,t)+\alpha vu_x(0,t)+\alpha mw_x(l,t)w_{xt}(l,t)\\&+\beta\rho l(w_t(l,t)+vw_x(l,t))^2-\beta\rho l(v+u_t(l,t)+vu_x(l,t))^2+\frac{J\alpha}{R^2}\left[\left(\frac{\alpha v}{EA}-u_t(0,t)\right)u_{tt}(0,t)-u_t(l,t)u_{tt}(l,t)\right]\Bigg\},\end{aligned} \tag{24}$$

$$\tau_2(t)=(\rho lR+2J/R)\dot{v}_d(t)-k_{\tau2}e(t)-\tau_1(t)+(T_{b1}-T_{b2})R, \tag{25}$$

where $k_{f1}(t)$ and $k_{f2}(t)$ will be determined later, and $k_{\tau1}$ and $k_{\tau2}$ are the positive control gains.

Remark 3. In implementing the control laws (23)–(25), the transverse displacement $w(l,t)$ can be measured with a displacement sensor attached to the hydraulic actuator. The slopes of the string, $w_x(0,t)$ and $w_x(l,t)$, can be obtained by using two laser sensors at each boundary [16]. The backward differencing of such signals can provide $w_t(l,t)$ and $w_{xt}(l,t)$. By multiplying the roller angle oscillations (measured by means of the encoders on the two rollers) by radius R, the longitudinal displacements at the boundaries, $u(0,t)$ and $u(l,t)$, can be determined [35]. Subsequently, the signals $u_t(0,t)$, $u_t(l,t)$, $u_{tt}(0,t)$, and $u_{tt}(l,t)$ can be obtained through the backward differencing of $u(0,t)$ and $u(l,t)$. Using the values of the material tension, $T(0,t)$ and $T(l,t)$, as measured by the tension sensors near the two rollers and the material tension formula (3), the values of $u_x(0,t)$ and $u_x(l,t)$ can be estimated. Finally, the transport velocity $v(t)$ can be measured with the tachometer on the right roller.

Preliminary to the stability analysis of the closed-loop system, four lemmas are established as follows.

Lemma 1. *If the initial tension T_0 and the two positive constants α and β in (20) satisfy the following inequalities*:

$$T_0 > \max(3\rho/2, EA(\sqrt{5}-2)/4), \tag{26}$$

$$\alpha > 2\beta l/[1-EA(\sqrt{5}-2)/(4T_0)], \tag{27}$$

then the following holds:

$$0 \le \gamma_1 W_1(t) \le V(t) \le \gamma_2 W_2(t), \tag{28}$$

where

$$W_1(t) = \int_0^l (u_t+vu_x)^2\,dx + \int_0^l u_x^2\,dx + \int_0^l (w_t+vw_x)^2\,dx + \int_0^l w_x^2\,dx + \int_0^l w_x^4\,dx + u^2(l,t)/2l$$
$$+u_t^2(0,t)+u_t^2(l,t)+w_x^2(l,t)+[w_t(l,t)+(v+2\beta l/\alpha)w_x(l,t)]^2+e^2(t), \tag{29}$$

$$W_2(t) = V_0(t), \tag{30}$$

$$\gamma_1 = \min\left(\frac{(\alpha-2\beta l)\rho}{2}, \frac{\alpha[1-EA(\sqrt{5}-2)/(4T_0)]-2\beta l}{4}, \frac{(\sqrt{5}-2)\alpha EA}{8\sqrt{5}}, \frac{\alpha}{2}, \frac{\alpha J}{2R^2}, \frac{\alpha m}{2}\right), \tag{31}$$

$$\gamma_2 = (1+\sqrt{5}/2)\alpha + 2\beta l. \tag{32}$$

Proof. Using the inequality $(a^2+b^2)/2 \ge ab$, we obtain

$$\frac{\alpha}{2}\int_0^l EAu_x w_x^2\,dx \le \frac{\alpha EA\sqrt{5}}{4}\int_0^l u_x^2\,dx + \frac{\alpha EA}{4\sqrt{5}}\int_0^l w_x^4\,dx, \tag{33}$$

$$2\beta\int_0^l \rho x u_x(v+u_t+vu_x)\,dx \le \beta\rho l\int_0^l u_x^2\,dx + \beta\rho l\int_0^l (v+u_t+vu_x)^2\,dx, \tag{34}$$

$$2\beta\int_0^l \rho x w_x(w_t+vw_x)\,dx \le \beta\rho l\int_0^l w_x^2\,dx + \beta\rho l\int_0^l (w_t+vw_x)^2\,dx. \tag{35}$$

Since $2u_x^2 \le w_x^2$ [28–30], we have

$$2\int_0^l u_x^2\,dx \le \int_0^l w_x^2\,dx. \tag{36}$$

Utilizing Eqs. (33)–(36), we arrive at

$$V(t) \le \alpha V_0(t) + \frac{\alpha\sqrt{5}}{4}\int_0^l EAu_x^2\,dx + \frac{\alpha}{4\sqrt{5}}\int_0^l EAw_x^4\,dx + \beta l\int_0^l \rho(v+u_t+vu_x)^2\,dx + \frac{3\beta l}{2}\int_0^l \rho w_x^2\,dx + \beta l\int_0^l \rho(w_t+vw_x)^2\,dx$$
$$\le [(1+\sqrt{5}/2)\alpha+2\beta l]V_0(t). \tag{37}$$

Similarly, we have

$$V(t) \ge \alpha V_0(t) - \frac{\alpha\sqrt{5}}{4}\int_0^l EAu_x^2\,dx - \frac{\alpha}{4\sqrt{5}}\int_0^l EAw_x^4\,dx - \frac{3\beta l}{2}\int_0^l \rho w_x^2\,dx - \beta l\int_0^l \rho(w_t+vw_x)^2\,dx - \beta l\int_0^l \rho(v+u_t+vu_x)^2\,dx$$
$$\ge \frac{(\alpha-2\beta l)\rho}{2}\int_0^l (v+u_t+vu_x)^2\,dx + \frac{\alpha[1-EA(\sqrt{5}-2)/(4T_0)]-2\beta l}{2}\int_0^l u_x^2\,dx + \frac{(\alpha-2\beta l)\rho}{2}\int_0^l (w_t+vw_x)^2\,dx$$
$$+\frac{\alpha[1-EA(\sqrt{5}-2)/(4T_0)]-2\beta l}{4}\int_0^l w_x^2\,dx + \frac{(\sqrt{5}-2)\alpha EA}{8\sqrt{5}}\int_0^l w_x^4\,dx + \frac{\alpha EA}{4l}u^2(l,t) + \frac{\alpha J}{2R^2}(u_t^2(0,t)+u_t^2(l,t))$$
$$+\frac{\alpha m}{2}w_x^2(l,t) + \frac{\alpha m}{2}[w_t(l,t)+(v+2\beta l/\alpha)w_x(l,t)]^2 + \frac{\alpha}{2}e^2(t) \ge \gamma_1 W_1(t). \tag{38}$$

The lemma is proved. □

Lemma 2. *The time derivative of the function $V(t)$ satisfies the inequality*

$$\dot{V}(t) \le -\lambda W_1(t), \tag{39}$$

where λ is a positive constant.

Proof. Differentiating Eq. (20) with respect to time yields

$$\dot{V}(t) = \alpha\dot{V}_0(t) + \alpha EA\int_0^l (w_{xt}+\upsilon w_{xx})u_x w_x\,dx + \frac{\alpha EA}{2}\int_0^l (u_{xt}+\upsilon u_{xx})w_x^2\,dx + \dot{V}_1(t). \tag{40}$$

Using Eqs. (9) and (10), $\dot{V}_0(t)$ is derived as

$$\begin{aligned}\dot{V}_0(t) = &\int_0^l (\upsilon+u_t+\upsilon u_x)(EA(u_x+w_x^2/2))_x\,dx + \int_0^l EA(u_{xt}+\upsilon u_{xx})u_x\,dx + \int_0^l (w_t+\upsilon w_x)((T_0+EA(u_x+w_x^2/2))w_x)_x\,dx\\ &+\int_0^l (w_{xt}+\upsilon w_{xx})(T_0+EAw_x^2/2)w_x\,dx - c_\upsilon\int_0^l (w_t+\upsilon w_x)^2\,dx + EAu(l,t)u_t(l,t)/2l + J(u_t(0,t)u_{tt}(0,t)+u_t(l,t)u_{tt}(l,t))/R^2\\ &+m[w_t(l,t)+(\upsilon+2\beta l/\alpha)w_x(l,t)][w_{tt}(l,t)+(\upsilon+2\beta l/\alpha)w_{xt}(l,t)]+e(t)\dot{e}(t).\end{aligned} \tag{41}$$

Furthermore, $\dot{V}_1(t)$ is obtained as

$$\begin{aligned}\dot{V}_1 \le\ &\beta\rho[x(u_t+\upsilon u_x)^2]_0^l + \beta EA[xu_x^2]_0^l - \beta\rho\int_0^l (u_t+\upsilon u_x)^2\,dx - \beta EA\int_0^l u_x^2\,dx + 2\beta EA\int_0^l xu_x w_x w_{xx}\,dx - (\beta\rho-\beta c_\upsilon l/\sigma_1)\int_0^l (w_t+\upsilon w_x)^2\,dx\\ &-(\beta T_0-\beta c_\upsilon l\sigma_1)\int_0^l w_x^2\,dx - \beta EA\int_0^l u_x w_x^2\,dx - \frac{\beta EA}{2}\int_0^l w_x^4\,dx + \beta EA\int_0^l xu_{xx}w_x^2\,dx\\ &+\beta EA\int_0^l xw_{xx}w_x^3\,dx + \beta lT(l,t)w_x^2(l,t) + \beta\rho l(w_t(l,t)+\upsilon w_x(l,t))^2.\end{aligned} \tag{42}$$

In deriving Eq. (42), Eqs. (9) and (10) and the following inequality were used:

$$2\int_0^l xw_x(w_t+\upsilon w_x)\,dx \le l\sigma_1\int_0^l w_x^2\,dx + \frac{l}{\sigma_1}\int_0^l (w_t+\upsilon w_x)^2\,dx, \tag{43}$$

where $\sigma_1 > 0$. Substituting Eqs. (41) and (42) into Eq. (40) and using the boundary conditions (11)–(14), Eq. (40) is rewritten as

$$\begin{aligned}\dot{V}(t) \le\ &-\beta\rho\int_0^l (\upsilon+u_t+\upsilon u_x)^2\,dx - (\beta\rho+(\alpha-\beta l)c_\upsilon/\sigma_1)\int_0^l (w_t+\upsilon w_x)^2\,dx - \beta EA(1-1/2k_{\tau1})\int_0^l u_x^2\,dx - \beta(T_0-c_\upsilon l\sigma_1-EA/2\sigma_2)\int_0^l w_x^2\,dx\\ &-\frac{\beta EA(3-2(\sigma_2+k_{\tau1}))}{4}\int_0^l w_x^4\,dx - \alpha\upsilon w_x^2(0,t)T(0,t) - \beta EAlw_x^4(0,t) + \beta EAlw_x^4(l,t) + \beta lT(l,t)w_x^2(l,t)\\ &+\alpha w_t(l,t)w_x(l,t)T(l,t) + \alpha\upsilon w_x^2(l,t)T(l,t) + \{\alpha w_t(l,t)+(\alpha\upsilon+2\beta l)w_x(l,t)\}\{f(t)-(c_a-\rho\upsilon)w_t(l,t)-T(x,t)w_x(l,t)\\ &+\rho\upsilon^2 w_x(l,t)+(\upsilon+2\beta l/\alpha)mw_{xt}(l,t)\} + \alpha(\upsilon+u_t(l,t)+\upsilon u_x(l,t))(u_x(l,t)+w_x^2(l,t)/2) + \frac{\alpha\upsilon\tau_1(t)}{EAR} - \frac{\alpha\upsilon(T_{b1}-T_0)}{EA} - \frac{\alpha\upsilon J}{EAR^2}u_{tt}(0,t)\\ &+\alpha\upsilon w_x^2(0,t)/2 - \alpha u_t(0,t)u_x(0,t) - \alpha\upsilon u_x^2(0,t) - \upsilon u_t(0,t)w_x^2(0,t)/2 - \alpha\upsilon u_x(0,t)w_x^2(0,t)/2 + EA\alpha u(l,t)u_t(l,t)/2l + \beta EAlu_x^2(l,t)\\ &+\beta\rho l(w_t(l,t)+\upsilon w_x(l,t))^2 + \alpha mw_x(l,t)w_{xt}(l,t) + \beta\rho l(\upsilon+u_t(l,t)+\upsilon u_x(l,t))^2 + J\alpha[u_t(0,t)u_{tt}(0,t)+u_t(l,t)u_{tt}(l,t)]/R^2 + e(t)\dot{e}(t).\end{aligned} \tag{44}$$

It is noted that the following inequalities were used in deriving Eq. (44)

$$\int_0^l xu_{xx}w_x^2\,dx + 2\int_0^l xu_x w_x w_{xx}\,dx \le [xu_x w_x^2]_0^l + \frac{1}{2k_{\tau1}}\int_0^l u_x^2\,dx + \frac{k_{\tau1}}{2}\int_0^l w_x^4\,dx, \tag{45}$$

$$-\int_0^l u_x w_x^2\,dx \le \frac{1}{2}\left(\frac{1}{\sigma_2}\int_0^l u_x^2\,dx + \sigma_2\int_0^l w_x^4\,dx\right), \tag{46}$$

where $\sigma_2 > 0$. Substituting Eqs. (24) and (25) into Eq. (18), we have

$$\dot{e}(t) = -\frac{k_{\tau2}R}{\rho lR^2+2J}e(t). \tag{47}$$

Substituting Eqs. (23), (24), and (47) into Eq. (44), we obtain

$$\begin{aligned}\dot{V}(t) \le\ &-\beta\rho\int_0^l (\upsilon+u_t+\upsilon u_x)^2\,dx - \left(\beta\rho+\frac{(\alpha-\beta l)c_\upsilon}{\sigma_1}\right)\int_0^l (w_t+\upsilon w_x)^2\,dx - \beta EA\left(1-\frac{1}{2k_{\tau1}}\right)\left(\int_0^l u_x^2\,dx + \frac{u^2(l,t)}{2l}\right)\\ &-\beta\left(T_0-c_\upsilon l\sigma_1-\frac{EA}{2\sigma_2}\right)\int_0^l w_x^2\,dx - \frac{\beta EA(3-2(\sigma_2+k_{\tau1}))}{4}\int_0^l w_x^4\,dx - \alpha\upsilon w_x^2(0,t)T(0,t) - \beta EAlw_x^4(0,t)\end{aligned}$$

$$-\alpha v EAu_x^2(0,t)-\alpha k_{f1}w_t^2(l,t)+[\alpha k_{f2}-k_{f1}(\alpha v+2\beta l)]w_t(l,t)w_x(l,t)$$
$$-[\beta l T_0-(\alpha v+2\beta l)k_{f2}]w_x^2(l,t)-\frac{1}{k_{\tau 1}}(u_t^2(0,t)+u_t^2(l,t))-\frac{k_{\tau 2}R}{\rho l R^2+2J}e^2(t). \tag{48}$$

Now, $k_{f1}(t)$, $k_{f2}(t)$, and $k_{\tau 1}$ are chosen to satisfy the following conditions:

$$k_{f1}(t)=\alpha\beta l T_0/2(\alpha v+2\beta l)^2, \tag{49}$$

$$k_{f2}(t)=-\beta l T_0/2(\alpha v+2\beta l), \tag{50}$$

$$3/2>k_{\tau 1}>1/2. \tag{51}$$

Since the value of T_0 is sufficiently large, there exist sufficiently large α, β, and sufficiently small σ_1 and σ_2 such that the conditions (26) and (27) and the following inequalities hold:

$$\eta_1=\beta\rho+(\alpha-\beta l)c_v/\sigma_1>0, \tag{52}$$

$$\eta_2=1-1/(2k_{\tau 1})>0, \tag{53}$$

$$\eta_3=T_0-c_v l\sigma_1-EA/(2\sigma_2)>0, \tag{54}$$

$$\eta_4=[3-2(\sigma_2+k_{\tau 1})]/4>0. \tag{55}$$

Then, the inequality

$$\dot V(t)\le-\lambda W_1(t)\le 0 \tag{56}$$

is obtained, where

$$\lambda=\min\left(\beta\rho,\eta_1,\beta EA\eta_2,\beta\eta_3,\beta EA\eta_4,\beta l T_0,\frac{\beta l T_0\alpha^2}{2(\alpha v+2\beta l)^2},\frac{1}{k_{\tau 1}},\frac{k_{\tau 2}R}{\rho l R^2+2J}\right). \tag{57}$$

The lemma thereby is proved. □

Lemma 3. (*Do and Pan* [9, p. 785]). *Given* $z(x,t):[0,l]\times\Re^+\to\Re$, *the following inequalities hold*:

$$\int_0^l z^2(x,t)\,\mathrm{d}x\le 2lz^2(0,t)+4l^2\int_0^l z_x^2(x,t)\,\mathrm{d}x, \tag{58}$$

$$\int_0^l z^2(x,t)\,\mathrm{d}x\le 2lz^2(l,t)+4l^2\int_0^l z_x^2(x,t)\,\mathrm{d}x, \tag{59}$$

$$z^2(x,t)<z^2(0,t)+2\sqrt{\int_0^l z^2(x,t)\,\mathrm{d}x}\sqrt{\int_0^l z_x^2(x,t)\,\mathrm{d}x}, \tag{60}$$

$$z^2(x,t)<z^2(l,t)+2\sqrt{\int_0^l z^2(x,t)\,\mathrm{d}x}\sqrt{\int_0^l z_x^2(x,t)\,\mathrm{d}x}. \tag{61}$$

Lemma 4. (*Fung et al.* [13, p. 437]). *If* $z(x,t):[0,l]\times\Re^+\to\Re$ *is uniformly bounded*, $\{z(x,t)\}_{x\in[0,l]}$ *is equi-uniformly continuous in t, and* $\lim_{t\to\infty}\int_0^t\|z(\tau)\|^2\,\mathrm{d}\tau$ *exists and is finite, then* $\lim_{t\to\infty}\|z(t)\|=0$.

Theorem 1. *Consider the system* (9) and (10) *with the boundary conditions* (11)–(14). $k_{f1}(t)$, $k_{f2}(t)$, $k_{\tau 1}$, and $k_{\tau 2}$ *in Eqs.* (23)–(25) *are selected to satisfy the conditions* (49)–(51)*. Then, the control laws* (23)–(25) *guarantee the uniform asymptotic convergence of the transverse and longitudinal vibrations and the velocity tracking error to zero.*

Proof. Consider the Lyapunov function candidate (20). Using Eq. (56), Lemmas 1, and 3, we have

$$\gamma_1 W_1(t)\le V(t)\le V(0)<\infty, \tag{62}$$

$$\frac{1}{4l^2}\int_0^l u^2\,\mathrm{d}x\le\frac{u^2(l,t)}{2l}+\int_0^l u_x^2\,\mathrm{d}x\le W_1(t)<\infty, \tag{63}$$

$$\frac{1}{4l^2}\int_0^l w^2\,\mathrm{d}x\le\int_0^l w_x^2\,\mathrm{d}x\le W_1(t)<\infty, \tag{64}$$

$$\int_0^l \dot u^2\,\mathrm{d}x=\int_0^l(u_t+vu_x)^2\,\mathrm{d}x\le W_1(t)<\infty, \tag{65}$$

$$\int_0^l \dot{w}^2\,\mathrm{d}x = \int_0^l (w_t + \nu w_x)^2\,\mathrm{d}x \le W_1(t) < \infty, \tag{66}$$

$$\int_0^l w_x^4\,\mathrm{d}x \le W_1(t) < \infty, \tag{67}$$

$$e^2(t) \le W_1(t) < \infty. \tag{68}$$

Define

$$\|u(x,t)\| = \left(\int_0^l u^2(x,t)\,\mathrm{d}x\right)^{1/2}, \tag{69}$$

$$\|w(x,t)\| = \left(\int_0^l w^2(x,t)\,\mathrm{d}x\right)^{1/2}. \tag{70}$$

Using Lemma 3 and Eqs. (63)–(66), we obtain

$$u^2 \le u^2(l,t) + 2\sqrt{\int_0^l u^2\,\mathrm{d}x}\sqrt{\int_0^l u_x^2\,\mathrm{d}x} < \infty, \tag{71}$$

$$w^2 \le 2\sqrt{\int_0^l w^2\,\mathrm{d}x}\sqrt{\int_0^l w_x^2\,\mathrm{d}x} < \infty, \tag{72}$$

$$\|u\|^2 + \|\dot{u}\|^2 + \|w\|^2 + \|\dot{w}\|^2 \le \frac{W_1(t)}{\gamma_3} < \infty, \quad \gamma_3 = \min\left(\frac{1}{4l^2}, 1\right), \tag{73}$$

$$\dot{V}(t) \le -\lambda\gamma_3\{\|u\|^2 + \|\dot{u}\|^2 + \|w\|^2 + \|\dot{w}\|^2\}. \tag{74}$$

From Eqs. (71) and (72), it can be concluded that u and w are bounded. Eq. (74) implies that

$$\int_0^\infty \|u\|^2\,\mathrm{d}t \le (V(0) - V(\infty))/(\lambda\gamma_3) < \infty, \tag{75}$$

$$\int_0^\infty \|w\|^2\,\mathrm{d}t \le (V(0) - V(\infty))/(\lambda\gamma_3) < \infty. \tag{76}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2) = \int_0^l 2u\dot{u}\,\mathrm{d}x \le \int_0^l (u^2 + \dot{u}^2)\,\mathrm{d}x \le \|u\|^2 + \|\dot{u}\|^2 < \infty, \tag{77}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2) = \int_0^l 2w\dot{w}\,\mathrm{d}x \le \int_0^l (w^2 + \dot{w}^2)\,\mathrm{d}x \le \|w\|^2 + \|\dot{w}\|^2 < \infty, \tag{78}$$

$\{u(x,t)\}_{x\in[0,l]}$ and $\{w(x,t)\}_{x\in[0,l]}$ are uniformly bounded and equi-uniformly continuous in t. Using Lemma 4, Eqs. (75) and (76) imply that $\lim_{t\to\infty}\|u(x,t)\| = 0$ and $\lim_{t\to\infty}\|w(x,t)\| = 0$, respectively. Using Eq. (56) and Lemma 1, we obtain

$$\dot{V}(t) \le -\lambda\gamma_3 e^2(t), \tag{79}$$

which implies

$$\int_0^\infty e^2(t)\,\mathrm{d}t \le \frac{(V(0) - V(\infty))}{\lambda\gamma_3} < \infty. \tag{80}$$

Using Eqs. (47) and (68), we have that $\dot{e}(t)$ is uniformly bounded. It follows from Barbalat's Lemma [36, p. 192] that $\lim_{t\to\infty}|e(t)| = 0$.

Since $W_1(t)$ is bounded, the boundedness of the total mechanical energy of the string system is obtained. From an engineering viewpoint that if the energy of the string system is bounded, then all the signals that constitute the governing dynamic equations will also be bounded, the following assumptions are made [3,14,16]: (i) If the kinetic energy of the string system (1) is bounded, then u_t, u_{xt}, w_t, and w_{xt} are bounded. (ii) If the potential energy of the string system (2) is bounded, then u_x, u_{xx}, w_x, and w_{xx} are bounded. Using Eq. (9) and the above statements, we have that u_{tt} is bounded. Since $e(t)$ and $\dot{e}(t)$ are bounded, Eqs. (17) and (19) imply that $\nu(t)$ and $\dot{\nu}(t)$ are bounded. Finally, it is concluded that the control laws (23)–(25) are bounded. □

4. Simulations

The finite difference method was employed to find an approximate solution for the PDEs with the boundary conditions (9)–(14). The convergence scheme is based on the central (for the string span) and forward/backward (for the left/right

Table 1
System parameters used in numerical simulation.

Parameter	Definition	Value
A	Cross-sectional area	0.4×0.001 m^2
E	Young's modulus	1.2×10^6 N/m^2
c_a	Damping coefficient of the damper	0.25 N m/s
c_v	Viscous damping coefficient of the string	0.001 N m^2 s
J	Moment of inertia of the roller	0.2 kg m^2
l	Distance between two rollers	4 m
m	Lumped mass of the hydraulic actuator	10 kg
R	Radius of the roller	0.1 m
T_0	Initial tension	1000 N
T_{b1}	Tension in the left adjacent span	2000 N
T_{b2}	Tension in the right adjacent span	400 N
ρ	Mass per unit length	2.7 kg/m

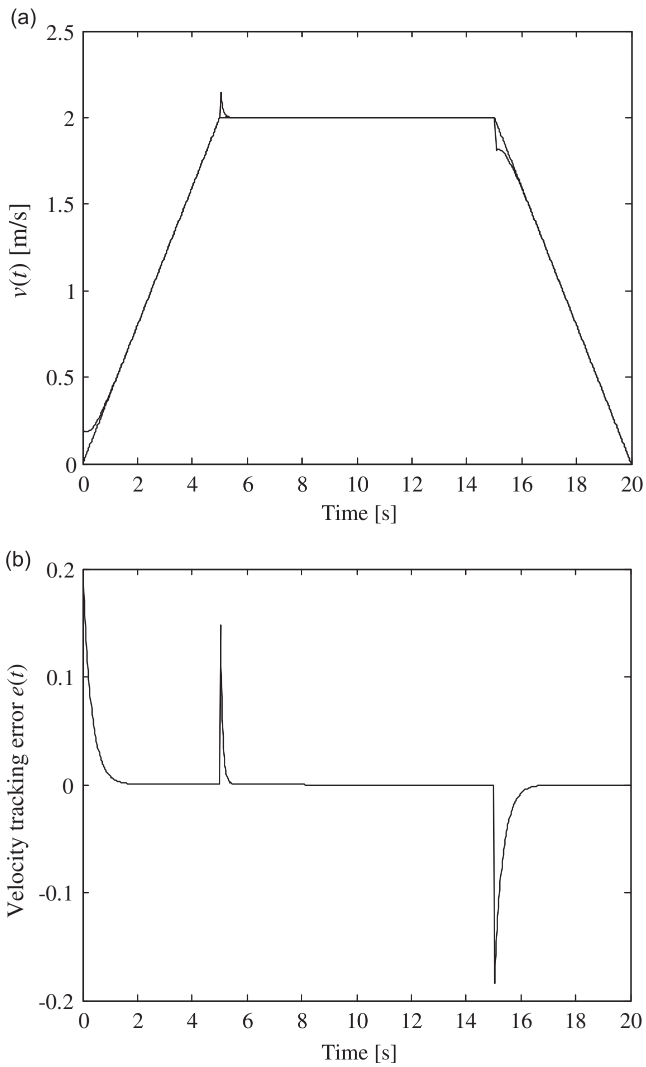

Fig. 2. (a) The regulated transport velocity (solid) obtained by Eq. (16) and the desired velocity profile (dotted) and (b) the velocity tracking error.

boundary) difference methods. The system parameters used in simulations are listed in Table 1. Let the initial conditions of the string be $u(x,0)=0.1\sin(\pi x/l)$, $u_t(x,0)=0$, $w(x,0)=0.5\sin(\pi x/l)$, and $w_t(x,0)=0$. In the simulation, the transport velocity was regulated to track a typical velocity profile (dotted line) widely used in practice, as shown in Fig. 2.

The dynamic responses of the axially moving string were simulated in two cases. In the first case, the longitudinal vibration suppression was not focused. The control force $f(t)$ was selected as (23), where $k_{f1}(t)$ and $k_{f2}(t)$ were given as (49) and (50). Since the torques $\tau_1(t)$ and $\tau_2(t)$ were used only to regulate the transport velocity, the control laws

$$\tau_1(t)=0, \tag{81}$$

$$\tau_2(t)=(\rho lR+2J/R)\dot{v}_d(t)-k_{\tau 2}e(t)+(T_{b1}-T_{b2})R, \tag{82}$$

were proposed, where $k_{\tau 1}=0$ and $k_{\tau 2}=40$. The positive constants α, β, σ_1, and σ_2 were chosen according to the inequalities (52), (54), and (55) as follows: $\alpha=10$, $\beta=0.1$, $\sigma_1=1$, and $\sigma_2=0.3$. In the second case, the proposed control scheme (23)–(25) for suppression of the longitudinal and transverse vibrations and regulation of the transport velocity was applied to the closed-loop system. The control gain $k_{\tau 1}=1.1$ was chosen. $k_{f1}(t)$, $k_{f2}(t)$, $k_{\tau 2}$, α, β, σ_1, and σ_2 were maintained as in the first case.

It should be noted that the velocity tracking error dynamics (47) is also obtained with the control torque laws (81) and (82). Therefore, the asymptotic convergence of the velocity tracking error to zero can be achieved in both simulation cases, as shown in Fig. 2. With regards to vibration suppression, good convergence of the longitudinal and transverse vibrations cannot be achieved with the control laws (23), (81), and (82), as shown in Figs. 3 and 4. In that case, as shown in Fig. 3a, the

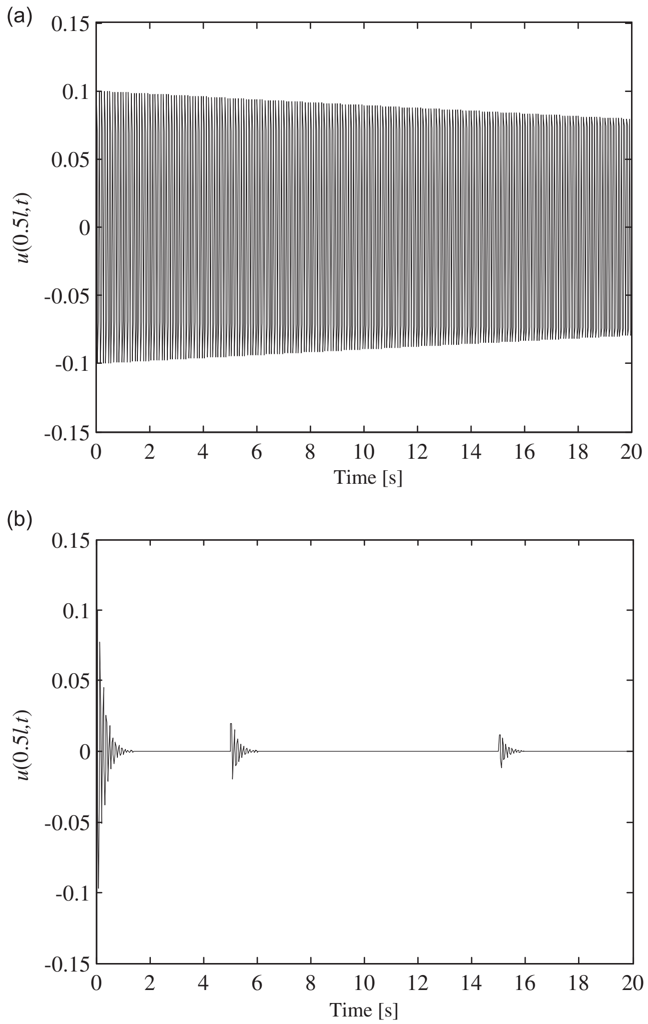

Fig. 3. Longitudinal displacements at $x=l/2$: (a) without longitudinal vibration suppression (the control laws (23), (81), and (82) are used) and (b) with longitudinal vibration suppression (the control laws (23)–(25) are used).

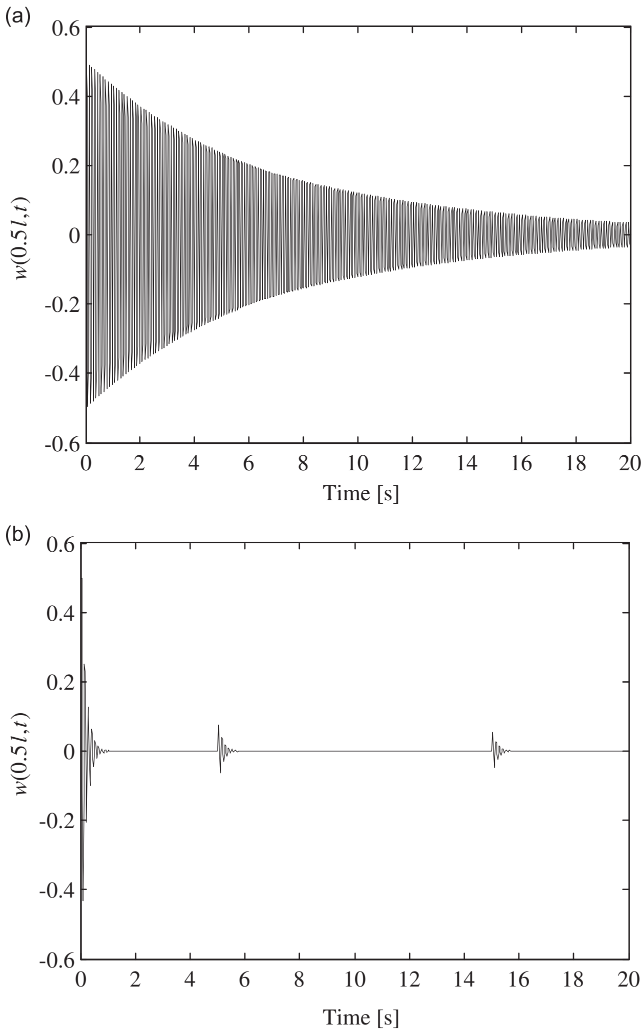

Fig. 4. Transverse displacements at $x=l/2$: (a) without longitudinal vibration suppression (the control laws (23), (81), and (82) are used) and (b) with longitudinal vibration suppression (the control laws (23)–(25) are used).

longitudinal vibration decreases very slowly; that is, when the transport velocity reaches zero at $t=20$, the value of the longitudinal displacement is not acceptable (i.e., $u(0.5l,20)=0.09$, 90 percent of the initial value). As mentioned in Remark 2, the longitudinal displacement affects the transverse displacement through the material tension. Therefore, when the longitudinal vibration cannot be suppressed, transverse vibration suppression with the control force (23) requires a great amount time: As shown in Fig. 4a, after 20 s, the value of the transverse displacement is still large (i.e., $w(0.5l,20)=0.05$, 10 percent of the initial value). As shown in Figs. 3b and 4b, the use of the proposed control scheme (23)–(25) provides the asymptotic convergence of both longitudinal and transverse vibrations; in other words, the longitudinal and transverse vibrations are suppressed completely within 1 s. As shown in Eqs. (9) and (10), the axial acceleration of the string $\dot{v}(t)$ affects to both the longitudinal and transverse dynamics. Therefore, when $\dot{v}(t)$ changes (at the beginning of a transport, at the transition point between acceleration and constant velocity, i.e., at 5 s in Fig. 2(a), and at the transition point between constant velocity and deceleration, i.e., at 15 s in Fig. 2(a)), the longitudinal and transverse vibrations occur, as shown in Figs. 3(b) and 4(b). Fig. 5 shows that the oscillation angle of the left roller ($u(0,t)/R$), the oscillation angle of the right roller ($u(l,t)/R$), and the hydraulic actuator displacement ($w(l,t)$) converge to zero. In Fig. 6, the control force and the two control torques are plotted.

5. Conclusions

In this paper, a control scheme for suppression of transverse and longitudinal vibrations and regulation of transport velocity was developed for an axially moving string system. The control scheme incorporates the control force exerted by

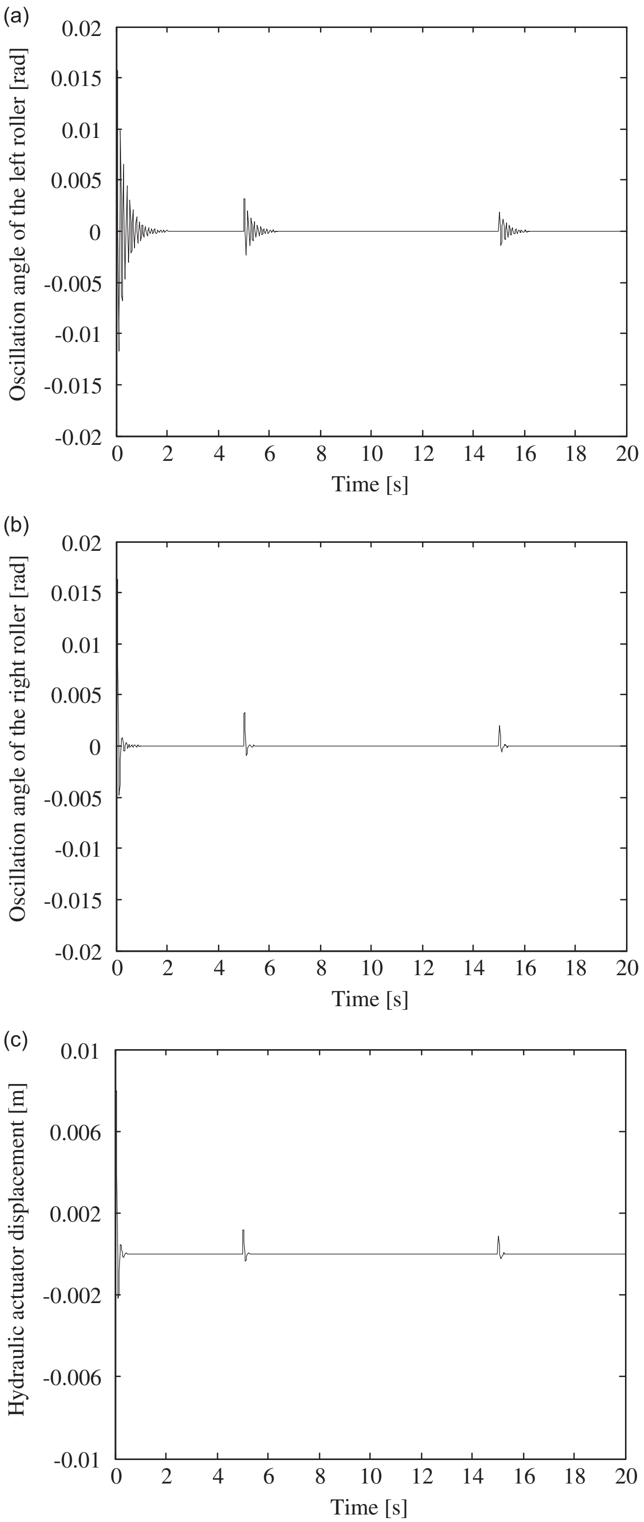

Fig. 5. Convergence of the motions of the actuators: (a) the oscillation angle of the left roller $u(0,t)/R$, (b) the oscillation angle of the right roller $u(l,t)/R$ and (c) the hydraulic actuator displacement $w(l,t)$.

the hydraulic actuator equipped with a damper at the right boundary and two control torques applied to the two drive rollers. With regards to the dynamics of the axially moving string, two nonlinear PDEs representing the longitudinal and transverse motions of the string, respectively, were derived with reference to the spatially varying tension. Based on the

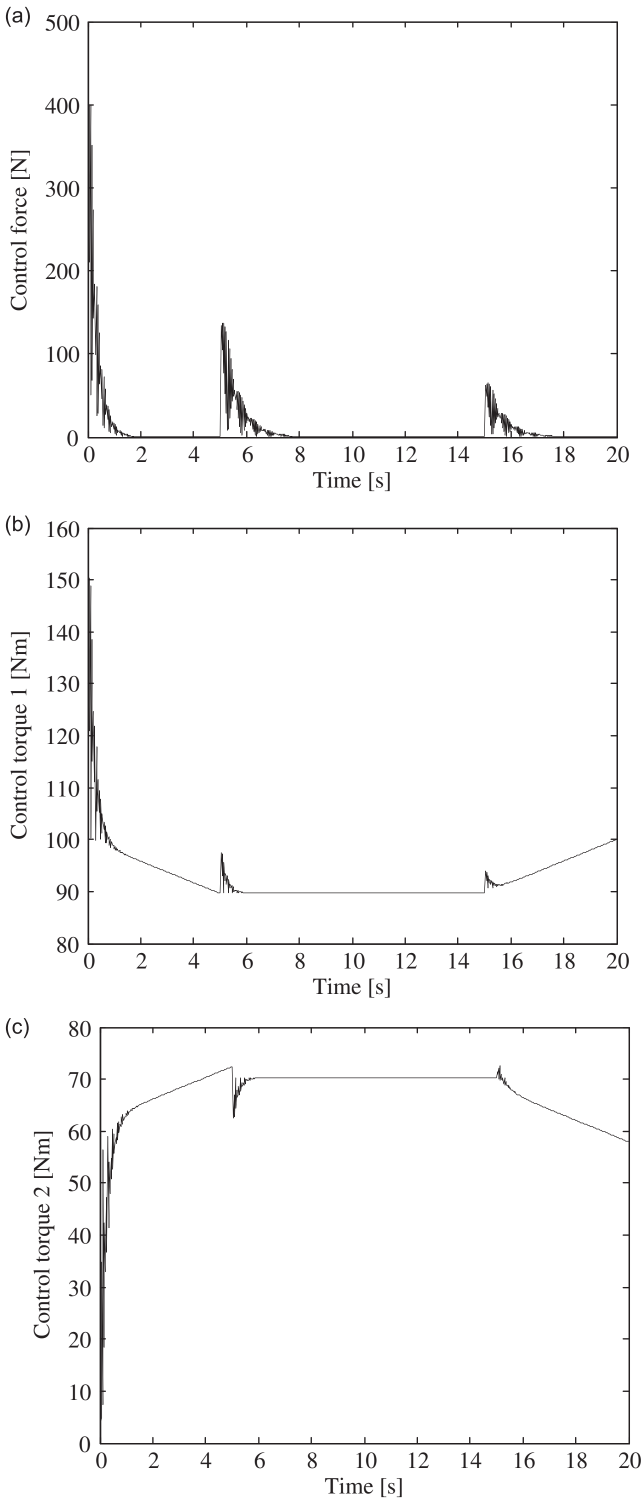

Fig. 6. The proposed control laws: (a) control force $f(t)$, (b) control torque $\tau_1(t)$, and (c) control torque $\tau_2(t)$.

energy of the axially moving string system, the Lyapunov method was employed to generate control laws. The stability of the closed-loop system, insofar as the longitudinal and transverse vibrations and velocity tracking error asymptotically converged to zero, was proved.

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References

[1] S.Y. Lee, C.D. Mote, Vibration control of an axially moving string by boundary control, *Journal of Dynamics System, Measurement, and Control—Transactions of the ASME* 118 (1) (1996) 66–74.
[2] W.D. Zhu, J. Ni, Energetics and stability of translating media with an arbitrary varying length, *Journal of Vibration and Acoustics—Transactions of the ASME* 122 (3) (2000) 295–304.
[3] S.P. Nagarkatti, F. Zhang, B.T. Costic, D.M. Dawson, Speed tracking and transverse vibration control of axially accelerating web, *Mechanical Systems and Signal Processing* 16 (2-3) (2002) 337–356.
[4] T. Li, Z. Hou, Exponential stabilization of an axially moving string with geometrical nonlinearity by linear boundary feedback, *Journal of Sound and Vibration* 296 (4–5) (2006) 861–870.
[5] T. Li, Z. Hou, J. Li, Stabilization analysis of a generalized nonlinear axially moving string by boundary velocity feedback, *Automatica* 44 (2) (2008) 498–503.
[6] Q.C. Nguyen, K.-S. Hong, Stabilization of an axially moving web via regulation of axial velocity, *Journal of Sound and Vibration* 330 (20) (2011) 4676–4688.
[7] Q.C. Nguyen, K.-S. Hong, Transverse vibration control of axially moving membranes by regulation of axial tension, *IEEE Transactions on Control Systems Technology* doi:10.1109/TCST.2011.2159384.
[8] Z. Qu, An iterative learning algorithm for boundary control of a stretched moving string, *Automatica* 38 (5) (2002) 821–827.
[9] K.D. Do, J. Pan, Boundary control of transverse motion of marine risers with actuator dynamics, *Journal of Sound and Vibration* 318 (4-5) (2008) 768–791.
[10] B.V. How, S.S. Ge, Y.S. Cho, Active control of flexible marine risers, *Journal of Sound and Vibration* 320 (4-5) (2009) 758–776.
[11] S.S. Ge, W. He, B.V.E. How, Y.S. Cho, Boundary control of a coupled nonlinear flexible marine riser, *IEEE Transactions on Control Systems Technology* 18 (5) (2010) 1080–1091.
[12] W. He, S.S. Ge, B.V.E. How, Y.S. Cho, K.-S. Hong, Robust adaptive boundary control of a flexible marine riser with vessel dynamics, *Automatica* 47 (4) (2011) 722–732.
[13] R.-F. Fung, J.W. Wu, S.L. Wu, Exponential stabilization of an axially moving string by linear boundary feedback, *Automatica* 35 (1) (1999) 177–181.
[14] F. Zhang, D.M. Dawson, S.P. Nagarkatti, Boundary control for a general class of non-linear string–actuator systems, *Journal of Sound and Vibration* 229 (1) (2000) 113–132.
[15] R.-F. Fung, J.W. Wu, P.Y. Lu, Adaptive boundary control of an axially moving system, *Journal of Vibration Acoustics—Transactions of the ASME* 124 (3) (2002) 435–440.
[16] Y. Li, D. Aron, C.D. Rahn, Adaptive vibration isolation for axially moving string: theory and experiment, *Automatica* 38 (3) (2002) 379–390.
[17] J.Y. Choi, K.-S. Hong, K.-J. Yang, Exponential stabilization of an axially moving tensioned strip by passive damping and boundary control, *Journal of Vibration and Control* 10 (5) (2004) 661–682.
[18] K.-J. Yang, K.-S. Hong, F. Matsuno, Robust adaptive boundary control of an axially moving string under a spatiotemporally varying tension, *Journal of Sound and Vibration* 273 (4-5) (2004) 1007–1029.
[19] K.-J. Yang, K.-S. Hong, F. Matsuno, Energy-based control of axially translating beams: varying tension, varying speed and disturbance adaptation, *IEEE Transactions on Control Systems Technology* 13 (6) (2005) 1045–1054.
[20] K.-J. Yang, K.-S. Hong, F. Matsuno, Robust boundary control of an axially moving string by using a PR transfer function, *IEEE Transactions on Automatic Control* 50 (12) (2005) 2053–2058.
[21] Q.C. Nguyen, K.-S. Hong, Asymptotic stabilization of a nonlinear axially moving string by adaptive boundary control, *Journal of Sound and Vibration* 329 (22) (2010) 4588–4603.
[22] H. Koc, D. Knittel, M. de Mathelin, G. Abba, Modeling and robust control of winding systems for elastic web, *IEEE Transactions on Control Systems Technology* 10 (2) (2002) 197–208.
[23] P.R. Pagilla, N.B. Siraskar, R.V. Dwivedula, Decentralized control of web processing lines, *IEEE Transactions on Control Systems Technology* 15 (1) (2007) 106–117.
[24] A. Benlatreche, D. Knittel, E. Ostertag, Robust decentralized control strategies for larger-scale web handling system, *Control Engineering Practice* 16 (6) (2008) 736–750.
[25] V. Gassmann, D. Kittel, P.R. Pagilla, M.-A. Bueno, Fixed-order *H*-infinity tension control in the unwinding section of a web handling system using a pendulum dance, *IEEE Transactions on Control Systems Technology* 20 (1) (2011) 173–180.
[26] S.P. Nagarkatti, F. Zhang, C.D. Rahn, D.M. Dawson, Tension and speed regulation for axially moving materials, *Journal of Dynamics System, Measurement, and Control—Transactions of the ASME* 122 (3) (2000) 445–453.
[27] H. Zhao, C.D. Rahn, Iterative learning velocity and tension control for single span axially moving materials, *Journal of Dynamics System, Measurement, and Control—Transactions of the ASME* 180 (5) (2008) 051003.1–051003.6.
[28] L.-Q. Chen, The energetic and the stability of axially moving strings undergoing planar motion, *International Journal of Engineering Science* 44 (18–19) (2006) 1346–1352.
[29] J. Chung, C.S. Han, K. Yi, Vibration of an axially moving string with geometric non-linearity and translating acceleration, *Journal of Sound and Vibration* 240 (4) (2001) 733–746.
[30] R. Narasimsha, Non-linear vibration of an elastic string, *Journal of Sound Vibration* 8 (1) (1967) 135–146.
[31] D.B. McIver, Hamilton's principle for systems of changing mass, *Journal of Engineering Mathematics* 7 (3) (1973) 249–261.
[32] C. Semler, G.X. Li, M.P. Paidousssis, The non-linear equations of motion of pipes conveying fluid, *Journal of Sound and Vibration* 169 (5) (1994) 577–599.
[33] H. Benaroya, T. Wei, Hamilton's principle for external viscous fluid–structure interaction, *Journal of Sound and Vibration* 238 (1) (2000) 113–145.
[34] U. Lee, I. Jang, On the boundary conditions for axially moving beams, *Journal of Sound and Vibration* 306 (3-5) (2007) 675–690.
[35] G. Michon, L. Manin, D. Dufour, R.G. Parker, Parametric instability of an axially moving belt subjected to multifrequency excitations: experiments and analytical validation, *Journal of Applied Mechanics—Transactions of the ASME* 75 (4) (2008) 041004.1–041004.8.
[36] H.K. Khalil, *Nonlinear Systems*, Prentice-Hall, New Jersey, USA, 1996.